Appendix 2.2

1. **Upper and Lower Bounds** If $\mathcal{S}$ is a set of real numbers then β is called an ***upper bound*** for $\mathcal{S}$ if

$$x \leq \beta \text{ for all } x \in \mathcal{S}.$$

Similarly, α is a ***lower bound*** for $\mathcal{S}$ if

$$\alpha \leq x \text{ for all } x \in \mathcal{S}.$$

We say that $\mathcal{S}$ is ***bounded*** if it has both an upper bound and a lower bound.

The ***least upper bound*** U for $\mathcal{S}$ satisfies

i) U is *an* upper bound for $\mathcal{S}$,

ii) $U \leq \beta$ for all upper bounds β of $\mathcal{S}$.

Equivalently,

i) U is *an* upper bound for $\mathcal{S}$,

ii') $\forall \varepsilon > 0, \exists\, a \in \mathcal{S} : U - \varepsilon < a \leq U$.

We can write either $U = \sup \mathcal{S}$ or $U = \operatorname{lub} \mathcal{S}$.

Think of the least upper bound as the least *of all* upper bounds for $\mathcal{S}$.

Similarly, the ***greatest lower bound***, L for $\mathcal{S}$ satisfies

i) L is *a* lower bound for $\mathcal{S}$,

ii) $L \geq \alpha$ for all lower bounds α of $\mathcal{S}$.

Equivalently,

i) L is *a* lower bound for $\mathcal{S}$,

ii') $\forall \varepsilon > 0, \exists\, a \in \mathcal{S} : L \leq a < L + \varepsilon$.

We can write either $L = \inf \mathcal{S}$ or $L = \operatorname{glb} \mathcal{S}$.

Think of the greatest lower bound as the greatest *of all* lower bounds for $\mathcal{S}$.

2. **Completeness Property of $\mathbb{R}$.**

   If a *non-empty* set $\mathcal{S} \subseteq \mathbb{R}$ has an *upper bound* then it has a *least* upper bound.

   If a *non-empty* set $\mathcal{S} \subseteq \mathbb{R}$ has a *lower bound* then it has a *greatest* lower bound.

   In the course MATH10242 these were called, respectively, the Upper Bound Axiom for $\mathbb{R}$ and the Lower Bound Axiom for $\mathbb{R}$.

   Assume only that non-empty sets bounded above have a least upper bound. Let $\mathcal{S}$ be a non-empty set bounded *below*, by α say. Then $-\alpha$ is an *upper* bound for the non-empty set $-\mathcal{S} := \{-s : s \in \mathcal{S}\}$. So, by our assumption, $-\mathcal{S}$ has a least upper bound U. It is not too hard to show that $-U$ is the greatest lower bound for $\mathcal{S}$. Thus we have shown that non-empty sets bounded below have a greatest lower bound. Thus the two parts of Completeness are **not** independent, each follows from the other.

3. $\mathbb{N}$ **is unbounded.** Perhaps you thought it obvious that $\mathbb{N}$ is an unbounded set, but how would you prove it? First, you can only prove something about a set such as $\mathbb{N}$ if you know its properties. In this case, one of the defining properties is that if $n \in \mathbb{N}$ then $n+1 \in \mathbb{N}$.

   Theorem 2.2.14 *Completeness of $\mathbb{R}$ implies that $\mathbb{N}$ is an unbounded set.*

   Proof by contradiction. Assume $\mathbb{N}$ is bounded above. Since $1 \in \mathbb{N}$ we have $\mathbb{N} \neq \varnothing$. Thus, by completeness of $\mathbb{R}$, the assumption that $\mathbb{N}$ is bounded means that $\kappa = \operatorname{lub} \mathbb{N}$ exists (but we are not saying it is in $\mathbb{N}$). By definition of lub, $\kappa - 1$ is no longer an upper bound for $\mathbb{N}$, and so there exists $n \in \mathbb{N}$ with $\kappa - 1 < n$. Rearrange as $\kappa < n+1$ and we have found an element of $\mathbb{N}$, namely $n+1$, strictly greater than an upper bound κ for $\mathbb{N}$. This contradiction means our assumption is false and $\mathbb{N}$ is an unbounded set. ■

4. **Archimedean Property** Perhaps you thought is obvious that given any $\alpha > 0$ you can find $n \in \mathbb{N}$ with $1/n < \alpha$, and we have certainly used this result earlier in the course without comment. But how would you prove it?

Definition 2.2.15 *The property*

$$\forall \alpha > 0, \exists n \in \mathbb{N} : 1/n < \alpha, \tag{5}$$

is called the ***Archimedean Property*** *of* $\mathbb{R}$.

Theorem 2.2.16 *The Archimedean Property of* $\mathbb{R}$ *holds if, and only if,* $\mathbb{N}$ *is an unbounded set.*

Proof

($\Rightarrow$) Assume that the Archimedean Property of $\mathbb{R}$ holds. Assume for contradiction that $\mathbb{N}$ is bounded above, by $\kappa > 0$ say. Take $\alpha = 1/\kappa$ in (5) to find $n \in \mathbb{N} : 1/n < \alpha = 1/\kappa$, i.e. $n > \kappa$. Thus we have found an element of $\mathbb{N}$, namely n, strictly greater than an upper bound κ for $\mathbb{N}$. Thus contradiction means our assumption is false and $\mathbb{N}$ is an unbounded set.

($\Leftarrow$) Assume that $\mathbb{N}$ is an unbounded set. Let $\alpha > 0$ be given. Then $1/\alpha$ is not an upper bound for $\mathbb{N}$ (since it doesn't have one) and so there exists $n \in \mathbb{N} : n > 1/\alpha$ which rearrange to $1/n < \alpha$. Thus we see that the Archimedean property holds. ■

5. The following lemma is simply an earlier one for limits rewritten for continuous functions. I give the proof again to help you remember it.

Lemma 2.2.17 *If* $g : A \to \mathbb{R}$ *is continuous at* $a \in \mathbb{R}$ *then*

i. if $g(a) > 0$ *there exists* $\delta > 0$ *such that* $|x - a| < \delta$ *implies* $g(x) > 0$,

ii. if $g(a) < 0$ *there exists* $\delta > 0$ *such that* $|x - a| < \delta$ *implies* $g(x) < 0$.

Proof left to Tutorial That g is continuous at a means $\lim_{x \to a} g(x) = g(a)$.

i. Assume $g(a) > 0$. Choose $\varepsilon = g(a)/2 > 0$ in the definition of limit to find $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < g(a)/2$. Open up as

$$-g(a)/2 < g(x) - g(a) < g(a)/2$$

and keep only the left hand inequality which rearranges as

$$g(x) > g(a) - g(a)/2 = g(a)/2 > 0.$$

ii. Assume $g(a) < 0$. Choose $\varepsilon = -g(a)/2$, which is > 0 since $g(a) < 0$. From the definition of limit we find $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < -g(a)/2$. Open up as

$$g(a)/2 < g(x) - g(a) < -g(a)/2$$

and keep only the right hand inequality which rearranges as

$$g(x) < g(a) - g(a)/2 = g(a)/2 < 0.$$

■

6. **Intermediate Value Theorem**. In previous years I gave a different proof. It starts with a lemma

Lemma 2.2.18 *If* $g : A \to \mathbb{R}$ *is continuous at* $a \in \mathbb{R}$ *then*

if $g(a) > 0$ *there exists* $\delta > 0$ *such that* $|x - a| < \delta$ *implies* $g(x) > 0$,

if $g(a) < 0$ *there exists* $\delta > 0$ *such that* $|x - a| < \delta$ *implies* $g(x) < 0$.

As in the notes we can reduce to a function g continuous on $[a, b]$, with $g(a) < 0 < g(b)$ and attempt to find $c \in (a, b) : g(c) = 0$.

Consider the set

$$\mathcal{S} = \{x \in [a, b] : g(x) < 0\}.$$

Then $\mathcal{S} \neq \phi$ since $a \in \mathcal{S}$, while $\mathcal{S} \subseteq [a, b]$ and so $\mathcal{S}$ is bounded above by b. Therefore, by the Completeness Axiom of $\mathbb{R}$ there exists $c \in [a, b] : c = \text{lub}\,\mathcal{S}$. As in the proof given we can assume $c \neq a$ or b, i..e. $c \in (a, b)$.

There are three possibilities $g(c) < 0$, $g(c) > 0$ and $g(c) = 0$.

Case 1: Assume $g(c) < 0$.

Continuity of g : By Lemma 2.2.18 there exists $\delta > 0$ such that if $|x - c| < \delta$ then $g(x) < 0$. If δ is sufficiently small then

$$\{x : |x - c| < \delta\} \subseteq [a, b] .$$

Consider the point $x_0 = c + \delta/2$. Then $x_0 \in \{x : |x - c| < \delta\}$ and so $g(x_0) < 0$, i.e $x_0 \in \mathcal{S}$, by the definition of $\mathcal{S}$. But $x_0 \in \mathcal{S}$ means $x_0 \leq c$ since c is an upper bound for $\mathcal{S}$. On the other hand $x_0 = c + \delta/2 > c$. Hence $x_0 \leq c$ and $x_0 > c$, a contradiction, which means our assumption is false and we do **not** have $g(c) < 0$.

Case 2 Assume $g(c) > 0$.

Continuity of g : By Lemma 2.2.18 there exists $\delta > 0$ such that if $|x - c| < \delta$ then $g(x) > 0$, that is $x \notin \mathcal{S}$.

Definition of c : The fact that $c = \operatorname{lub} \mathcal{S}$ means that c is the *least* of all upper bounds for $\mathcal{S}$ in which case $c - \delta$, with the δ just found, is **not** an upper bound for $\mathcal{S}$. In turn this means there exists $x_1 \in \mathcal{S}$ and $c \geq x_1 > c - \delta$.

But $x_1 \in \mathcal{S}$ means $g(x_1) < 0$ while $c \geq x_1 > c - \delta$, written as $|x_1 - c| < \delta$ implies $g(x) > 0$. Hence $g(x_1) < 0$ and $g(x) > 0$, a contradiction, which means our assumption is false and we do **not** have $g(c) < 0$.

Case 3 Because every other possibility leads to a contradiction we must therefore have $g(c) = 0$. ■

7. **Intermediate Value Theorem** The contradictions in cases 1 and 2 can be given in many different ways. In previous editions of these notes I gave the following

   Case 1: Assume $g(c) < 0$.

   Continuity of g : By Lemma 2.2.18 there exists $\delta > 0$ such that if $|x - c| < \delta$ then $g(x) < 0$. If δ is sufficiently small then

   $$\{x : |x - c| < \delta\} \subseteq [a, b] .$$

   This, with $g(x) < 0$, means that

   $$\{x : |x - c| < \delta\} \subseteq \mathcal{S}.$$

This means that $\text{lub}\,\mathcal{S} \ge c + \delta > c = \text{lub}\,\mathcal{S}$ by the definition of c. Hence $\text{lub}\,\mathcal{S} > \text{lub}\,\mathcal{S}$ with *strict* inequality. This contradiction means our assumption is false and we do **not** have $g(c) < 0$.

Case 2 Assume $g(c) > 0$.

Continuity of g : By Lemma 2.2.18 there exists $\delta > 0$ such that if $|x - c| < \delta$ then $g(x) > 0$, that is $x \notin \mathcal{S}$.

Definition of c : The fact that $c = \text{lub}\,\mathcal{S}$ means that c is the *least* of all upper bounds for $\mathcal{S}$ in which case $c - \delta$, with the δ just found, is **not** an upper bound for $\mathcal{S}$. In turn this means there exists $x_1 \in \mathcal{S}$ and $c \ge x_1 > c - \delta$.

So we have both

$$c - \delta < x < c + \delta \implies x \notin \mathcal{S} \quad \text{and} \quad \exists\, x_0 : c - \delta < x_0 < c,\; x_0 \in \mathcal{S}.$$

This contradiction at x_0 means our assumption is false and we do **not** have $g(c) > 0$.

8. **Theorem 2.2.19** ***The Bolzano-Weierstrass Theorem (1817)*** *A bounded infinite sequence of real numbers has a convergent subsequence.*

   This proof makes use of the results that an increasing sequence of real numbers bounded above is convergent, and a decreasing sequence of real numbers bounded below is convergent. In turn these follow from the Completeness of the Real numbers, the limits of these sequences being the lub and glb respectively of the sequences considered as sets.

   Proof Suppose the infinite sequence $\{\gamma_n\}_{n\ge1}$ lies in the interval $[a, b]$. We construct a convergent subsequence by a bisection process.

   Split the interval $[a, b]$ into two halves $[a, c]$ and $[c, b]$, so $c = (a + b)/2$. Then (at least) one of the halves will contain infinitely many terms of the sequence. Choose that interval, though if both contain infinitely many choose the left interval. Label the sub-interval as $[a_1, b_1]$ and note that $a \le a_1 < b_1 \le b$. Choose γ_{n_1} to be any point of the sequence in $[a_1, b_1]$.

   Then split $[a_1, b_1]$ in half again and repeat the process choosing $[a_2, b_2]$ to contain infinitely many sequence points and γ_{n_2} to be one such point with $n_2 > n_1$. Note that $a_1 \le a_2 < b_2 \le b_1$

Continuing in this way, at the k-th stage we will choose an interval $[a_k, b_k]$, where $a_{k-1} \leq a_k < b_k \leq b_{k-1}$, and a term $\gamma_{n_k} \in [a_k, b_k]$ with $n_k > n_j$ for all $j < k$. Thus we have a subsequence $\{\gamma_{n_k}\}_k$ satisfying $n_1 < n_2 < n_3 < ...$ and $a_k \leq \gamma_{n_k} \leq b_k$ for all $k \geq 1$.

Note that because of the halving process

$$b_k - a_k = \frac{1}{2}(b_{k-1} - a_{k-1})$$

which can be continued to give

$$b_k - a_k = \left(\frac{1}{2}\right)^k (b - a) \tag{6}$$

for all $k \geq 1$.

The sequence $\{a_n\}_n$ of the left-hand ends of intervals $[a_n, b_n]$ is monotonic increasing, bounded above by b and hence has a limit α, say.

The sequence $\{b_n\}_n$of right-hand ends of intervals is monotonic decreasing, bounded below by a and hence has a limit β, say

The inequalities

$$a_{k-1} \leq a_k < b_k \leq b_{k-1} \quad \text{for all } k \geq 1$$

imply that

$$a_n < b_m \quad \text{for all } m, n \geq 1.$$

Fix $m \geq 1$. Then we have $a_n < b_m$ for all $n \geq 1$ which implies that $\lim_{n\to\infty} a_n \leq b_m$, i.e. $\alpha \leq b_m$. But this is true for all $m \geq 1$ which implies that $\alpha \leq \lim_{m\to\infty} b_m$, i.e. $\alpha \leq \beta$.

The inequalities $a_{k-1} \leq a_k < b_k \leq b_{k-1}$ for all $k \geq 1$ also imply $a_m \leq a_k < b_k \leq b_m$ for all $k \geq m \geq 1$. Fix $m \geq 1$ and let $k \to \infty$ to get $a_m \leq \alpha < \beta \leq b_m$. Thus

$$0 \leq \beta - \alpha \leq b_m - a_m = \left(\frac{1}{2}\right)^m (b - a),$$

by (6). Let $m \to \infty$ to deduce that $\alpha = \beta$ called ℓ. Returning to

$$a_k \leq \gamma_{n_k} \leq b_k,$$

let $k \to \infty$ to deduce, by the Sandwich Rule, that $\lim_{k\to\infty} \gamma_{n_k} = \ell$. ■

9. **Compact Sets** The proof of the result that a function continuous on a closed, bounded interval, $[a,b]$, is bounded was quite long. You will come across the result again in Topology. But you will come across it as a real-valued continuous function on a *compact* set is bounded. We have not discussed compactness, though a closed bounded interval of $\mathbb{R}$ is compact. Using the properties of compactness will shorten the proof of the boundedness result substantially.

10. **Lower Bound part of Boundedness Theorem**

    Theorem 2.2.20 ***Boundedness Theorem*** *(1861) Suppose that f is a function continuous on a closed and bounded interval $[a,b]$. Then there exist $c, d \in [a,b]$ such that*

    $$f(c) \le f(x) \le f(d)$$

    for all $x \in [a,b]$.

    Proof In the lectures I gave the proof that the upper bound is attained. I suggest that you write out the proof for the lower bound. You can now check if your proof is correct:

    By the previous Theorem f is bounded on $[a,b]$. Look at the lower bounds and set $m = \text{glb}_{x\in[a,b]} f(x)$.

    Use proof by contradiction, so assume the lower bound is not attained, i.e. there does **not** exist $c \in [a,b]$ with $f(c) = m$. Thus so $f(x) > m$ for all $x \in [a,b]$. Define

    $$g(x) = \frac{1}{f(x) - m},$$

    well-defined since $f(x) \ne m$ for all $x \in [a,b]$. By the quotient rule this is continuous on $[a,b]$ and so, by the previous Theorem it is bounded. Thus there exists $L > 0$ such that $g(x) \le L$. Rearrange to get

    $$f(x) > m + \frac{1}{L}$$

    for all $x \in [a,b]$. But this means that $m + 1/L$ is *a* lower bound for f. Yet m is the *greatest* of all lower bounds. This is a contradiction, so the assumption is false, i.e. the lower bound is attained at some point. ■

11. **An alternative proof that a continuous function on a closed and bounded interval attains its bounds** can be given using sequences.

Since M is the *least* of all upper bounds then, for each $n \geq 1$, $M - 1/n$ is **not** an upper bound for $\{f(x) : a \leq x \leq b\}$ and so you can find $a \leq x_n \leq b$ such that $f(x_n) > M - 1/n$.

Let $\{x_{n_k}\}_{k \geq 1}$ be a convergent subsequence of $\{x_n\}_{n \geq 1}$, which exists by Bolzano-Weierstrass, and set $d = \lim_{k \to \infty} x_{n_k}$.

Let $\varepsilon > 0$ be given. Choose $N : 1/N < \varepsilon$ (by the Archimedean Property of $\mathbb{R}$ discussed above). Then for all $k \geq N$ we have

$$\begin{aligned} M &\geq f(x_{n_k}) > M - \frac{1}{n_k} \geq M - \frac{1}{k} \quad \text{since } n_k \geq k \\ &\geq M - \frac{1}{N} \geq M - \varepsilon. \end{aligned}$$

That is for all $k \geq N$, $|f(x_{n_k}) - M| < \varepsilon$. This is the definition of $\lim_{k \to \infty} f(x_{n_k}) = M$. Yet by continuity, $\lim_{k \to \infty} f(x_{n_k}) = f(d)$ hence $f(d) = M$.